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# Approximate Analytical Solution of Nonlinear Evolution Equations

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## Abstract

Analytical solitary wave solution of the dust ion acoustic waves (DIAWs) is studied in the frame-work of Korteweg-de Vries (KdV), damped force Korteweg-de Vries (DFKdV), damped force modified Korteweg-de Vries (DFMKdV) and damped forced Zakharov-Kuznetsov (DFZK) equations in an unmagnetized collisional dusty plasma consisting of negatively charged dust grain, positively charged ions, Maxwellian distributed electrons and neutral particles. Using reductive perturbation technique (RPT), the evolution equations are obtained for DIAWs.

**Keywords:** solitary wave, soliton, KdV, DKdV, DFZK

## 1. Introduction

In the field of physics and applied mathematics research getting an exact solution of a nonlinear partial differential equation is very important. The elaboration of many complex phenomena in fluid mechanics, plasma physics, optical fibers, biology, solid-state physics, etc. is possible if analytical solutions can be obtained. Most of the differential equation arises in these field has no explicit solution as popularly known. This problem creates hindrances in the study of nonlinear phenomena and makes it time-consuming in the research of nonlinear models in the plasma and other science. However recent researches in nonlinear differential equations have seen the development of many approximate analytical solutions of partial and ordinary differential equations.

The history behind the discovery of soliton is not only interesting but also significant. In 1834 a Scottish scientist and engineer—John Scott-Russell first noticed the solitary water wave on the Edinburgh Glasgow Canal. In 1844 [1] in “Report on Waves” he accounted his examinations to the British Association. He wrote “I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834 was my first

chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.” He coined the word “solitary wave.” The solitary wave is called so because it often occurs as a single entity and is localized. The most important characteristics of solitary waves were unearthed after thorough study along with extensive wave-tank experiments. The following are the properties of solitary waves:

(a) These localized bell-shaped waves travel with enduring form and velocity. The speed of these waves are given by  $c^2 = g(h + a)$ , where  $g$ ,  $a$ ,  $h$  are respectively represent the acceleration of the gravity, amplitude of the wave and the undisturbed depth of the water. (b) Solitary waves can cross each other without any alteration.

John Scott-Russell’s study created a stir in the scientific community. His study not only initiated a debate with the prevailing knowledge of the theories of waves but also challenged the antecedent knowledge of waves. The previous study claimed that a periodic wave of finite amplitude and permanent shape are feasible only in deep water unlike Russell’s observation that the permanent profile is also possible in shallow water. Finally the stable form of solitary waves was received in scientific community with the aid of nonlinearity and dispersion. An ideal equilibrium between nonlinearity and dispersion can generate such waves.

Diederik Johannes Korteweg in 1895 [2] along with his PhD student Gustav De Vries obtained an equation from the primary equation of hydrodynamics. This equation explains shallow water waves where the existence of solitary waves was mathematically recognized. This equation is called KdV equation which is of the form  $\frac{\partial u}{\partial t} + Au \frac{\partial u}{\partial x} + B \frac{\partial^3 u}{\partial x^3} = 0$ . One of the most popular equations of soliton theory, this equation helps in explaining primary ideas that lie behind the soliton concept. Martin Zabusky and Norman Kruskal [3] in 1965 solved KdV equation numerically and noticed that the localized waves retain their shape and momentum in collisions. These waves were known as “solitons.” Soliton are solitary waves with the significant property that the solitons maintain the form asymptotically even when it experiences a collision. The fundamental “microscopic” properties of the soliton interaction; (i) the interaction does not change the soliton amplitudes; (ii) after the interaction, each soliton gets an additional phase shift; (iii) the total phase shift of a soliton acquired during a certain time interval can be calculated as a sum of the elementary phase shifts in pair wise collisions of this soliton with other solitons during this time interval is of importance. Solitons are mainly used in fiber optics, optical computer etc. which has really generated a stir in today’s scientific community. The conventional signal dispensation depends on linear system and linear systems. After all in this case nonlinear systems create more well-organized algorithms. The optical soliton is comparatively different from KdV solitons. Unlike the KdV soliton that illustrates the wave in a solitary wave, the optical soliton in fibers is the solitary wave of an envelope of a light wave. In this regard, the optical soliton in a fiber is treated as an envelope soliton.

This chapter will discuss the analytical solitary wave solution of the KdV and KdV-like equations. In the study of nonlinear dispersive waves, these equations are generally seen. The KdV equation, a generic equation, is important in the study of weakly nonlinear long waves. This equation consists of a single humped wave characterized by several unique properties. The Soliton solutions of the KdV equation have been quite popular but it also not devoid of problems. The problems not only restrict to dispersion but also dissipation and interestingly these are not dominated by the KdV equation. The standard KdV equation fails to explain the development of small-amplitude solitary waves in case the particles collide in a plasma system. KdV equation with an additional damping term or the damped

Korteweg-de Vries (DKdV) equation becomes handy in explaining this issue of elaborating the character of the wave. But in the presence of any critical physical situation (critical point) nonlinearity of the KdV equation disappears and the amplitude of the waves reaches infinity. To control this situation, a new nonlinear partial differential equation has to be derived that can explain the system at that critical point. This is known as the modified Korteweg-de Vries (MKdV) equation. In the presence of collisions, this equation is not also adequate and a damped MKdV equation is necessary. Also in the presence of force source term then the equation will be further modified and become DFKdV/DFMKdV.

## 2. The Korteweg-de Vries equation

Now we will derive the KdV equation from a classic plasma model, in which we consider a collision-free unmagnetized plasma consists of electrons and ions, in which ions are mobile and electrons obey the Maxwell distribution. The basic equation will be given as:

$$\frac{\partial N_i}{\partial T} + \frac{\partial N_i U_i}{\partial X} = 0 \quad (1)$$

$$\frac{\partial U_i}{\partial T} + U_i \frac{\partial U_i}{\partial X} = -\frac{e}{m_i} \frac{\partial \psi}{\partial X} \quad (2)$$

$$\epsilon_0 \frac{\partial^2 \psi}{\partial X^2} = e(N_e - N_i) \quad (3)$$

where the electrons obey Maxwell distribution, i.e.,  $N_e = en_0 e^{\frac{e\phi}{K_B T_e}}$ .  $N_i$ ,  $N_e$ ,  $U_i$ ,  $m_i$  are the ion density, electron density, ion velocity and ion mass, respectively.  $\psi$  is the electrostatic potential,  $K_B$  is the Boltzmann constant,  $T_e$  is the electron temperature and  $e$  is the charge of the electrons.

To write Eqs. (1)–(3) in dimensionless form we introduce the following dimensionless variables

$$x = \frac{X}{\lambda_D}, t = \omega_p T, \phi = \frac{e\psi}{K T_e}, n_i = \frac{N_i}{n_0}, u_i = \frac{U_i}{c_s}, \quad (4)$$

where  $\lambda_D = \sqrt{\epsilon_0 K_B T_e / n_0 e^2}$  is the Debye length,  $c_s = \sqrt{K_B T_e / m_i}$  is the ion acoustic speed,  $\omega_{pi} = \sqrt{n_0 e^2 / \epsilon_0 m_i}$  is the ion plasma frequency and  $n_0$  is the unperturbed density of ions and electrons. Hence using (4) in (1)–(3) we obtain the normalized set of equations as

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i u_i)}{\partial x} = 0 \quad (5)$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{\partial \phi}{\partial x} \quad (6)$$

$$\frac{\partial^2 \phi}{\partial x^2} = e^\phi - n_i \quad (7)$$

To linearized (5)–(7), let us write the dependent variable as sum of equilibrium and perturbed parts, so that we write  $n_i = 1 + \bar{n}_i$ ,  $u_i = \bar{u}_i$ ,  $\phi = \bar{\phi}$ . Putting  $n_i = 1 + \bar{n}_i$  where the values of parameters at equilibrium position is given by  $n_1 = 1$ ,  $u_1 = 0$  and  $\phi_1 = 0$  in Eq. (5), we get

$$\frac{\partial}{\partial t}(1 + \bar{n}_i) + \frac{\partial}{\partial x}(\bar{u}_i + \bar{n}_i \bar{u}_i) = 0 \quad (8)$$

neglecting the nonlinear term  $\frac{\partial(\bar{n}_i \bar{u}_i)}{\partial x}$  from (8), we get

$$\frac{\partial \bar{n}_i}{\partial t} + \frac{\partial \bar{u}_i}{\partial x} = 0 \quad (9)$$

which is the linearized form of Eq. (5).

Putting  $u_i = \bar{u}_i$ ,  $\phi = \bar{\phi}$  in Eq. (6), we get

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_i \frac{\partial \bar{u}_i}{\partial x} = -\frac{\partial \bar{\phi}}{\partial x} \quad (10)$$

Neglecting the nonlinear term from (10), we get

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{\phi}}{\partial x} = 0 \quad (11)$$

This is the linearized form of Eq. (6).

Putting  $n_i = 1 + \bar{n}_i$ ,  $\phi = \bar{\phi}$  in Eq. (7), we get

$$\begin{aligned} \frac{\partial^2 \bar{\phi}}{\partial x^2} &= 1 + \bar{\phi} - 1 - \bar{n}_i \\ \Rightarrow \frac{\partial^2 \bar{\phi}}{\partial x^2} &= \bar{\phi} - \bar{n}_i \end{aligned} \quad (12)$$

Hence Eqs. (9), (11), (12) are the linearized form of Eq. (5)–(7) respectively.

To get dispersion relation for low frequency wave let us assume that the perturbation is proportional to  $e^{i(kx - \omega t)}$  and of the form

$$\bar{n} = n_0 e^{i(kx - \omega t)} \quad (13)$$

$$\bar{u} = u_0 e^{i(kx - \omega t)} \quad (14)$$

$$\bar{\phi} = \phi_0 e^{i(kx - \omega t)} \quad (15)$$

So,

$$\frac{\partial \bar{n}}{\partial t} = -in_0 \omega e^{i(kx - \omega t)} \quad (16)$$

$$\frac{\partial \bar{n}}{\partial x} = ikn_0 e^{i(kx - \omega t)} \quad (17)$$

$$\frac{\partial \bar{u}}{\partial t} = -iu_0 \omega e^{i(kx - \omega t)} \quad (18)$$

$$\frac{\partial \bar{u}}{\partial x} = iku_0 e^{i(kx - \omega t)} \quad (19)$$

$$\frac{\partial \bar{\phi}}{\partial x} = ik\phi_0 e^{i(kx - \omega t)} \quad (20)$$

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} = (ik)^2 \phi_0 e^{i(kx - \omega t)} \quad (21)$$

Putting these value in Eqs. (9), (11) and (12), we get,

$$-i\omega n_0 + iku_0 = 0 \quad (22)$$

$$-i\omega u_0 + ik\phi_0 = 0 \quad (23)$$

$$n_0 - (k^2 + 1)\phi_0 = 0 \quad (24)$$

Since the system (22)–(24) is a system of linear homogeneous equation so for nontrivial solutions we have

$$\begin{vmatrix} -i\omega & ik & 0 \\ 0 & -i\omega & ik \\ 1 & 0 & -(k^2 + 1) \end{vmatrix} = 0 \quad (25)$$

$$\Rightarrow -i^2\omega^2(k^2 + 1) + i^2k^2 = 0$$

$$\Rightarrow \omega^2(k^2 + 1) = -i^2k^2$$

$$\Rightarrow \omega^2 = \frac{k^2}{(k^2 + 1)}$$

This is the **dispersion relation**.

For small  $k$ , i.e., for weak dispersion we can expand as

$$\begin{aligned} \omega &= k(1 + k^2)^{-\frac{1}{2}} \\ &= k - \frac{1}{2}k^3 + \dots \end{aligned} \quad (26)$$

The phase velocity as

$$V_p = \frac{\omega}{k} = \frac{1}{\sqrt{1 + k^2}} \quad (27)$$

so that  $V_p \rightarrow 1$  as  $k \rightarrow 0$  and  $V_p \rightarrow 0$  as  $k \rightarrow \infty$ . The group velocity  $V_g = \frac{d\omega}{dk}$  is given by

$$V_g = \frac{1}{(1 + k^2)^{3/2}} \quad (28)$$

In this case, we have  $V_g < V_p$  for all  $k > 0$ . The group velocity is more important as energy of a medium transfer with this velocity.

For long-wave as  $k \rightarrow 0$ , the leading order approximation is  $\omega = k$ , corresponding to non-dispersive acoustic waves with phase speed  $\omega/k = 1$ . Hence this speed is the same as the speed of the ion-acoustic waves  $c_s$ . The long wave dispersion is weak, i.e.,  $k\lambda_D \ll 1$ . This means that the wavelength is much larger than the Debye length. In these long waves, the electrons oscillate with the ions. The inertia of the wave is provided by the ions and the restoring pressure force by the electrons. At the next order in  $k$ , we find that

$$\omega = k - \frac{1}{2}k^3 + O(k^5) \quad \text{as } k \rightarrow 0 \quad (29)$$



The  $O(k^5)$  correction corresponds to weak KdV type long wave dispersion. For short wave ( $k \rightarrow \infty$ ), the frequency  $\omega = 1$ , corresponding to the ion plasma frequency  $\omega_{pi} = \frac{c_s}{\lambda_D}$ . Hence the ions oscillate in the fixed background of electrons.

Now the phase of the waves can be written as

$$kx - \omega t = k(x - t) + \frac{1}{2}k^3 t \quad (30)$$

Here  $k(x - t)$  and  $k^3 t$  have same dynamic status (dimension) in the phase. Assuming  $k$  to be small order of  $\varepsilon^{1/2}$ ,  $\varepsilon$  being a small parameter measuring the weakness of the dispersion, Here  $(x - t)$  is the traveling wave form and time  $t$  is the linear form.

Let us consider a new stretched coordinates  $\xi, \tau$  such that

$$\xi = \varepsilon^{1/2}(x - \lambda t), \tau = \varepsilon^{3/2}t \quad (31)$$

where  $\varepsilon$  is the strength of nonlinearity and  $\lambda$  is the Mach number (phase velocity of the wave).  $\varepsilon$  may be termed as the size of the perturbation. Let the variables be perturbed from the stable state in the following way (considering  $n_i = 1, u_i = 0, \phi = 0$  and  $n_e = e^\phi = e^0 = 1$  at equilibrium)

$$n_i = 1 + \varepsilon n_i^{(1)} + \varepsilon^2 n_i^{(2)} + \varepsilon^3 n_i^{(3)} + \dots, \quad (32)$$

$$u_i = 0 + \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)} + \varepsilon^3 u_i^{(3)} + \dots, \quad (33)$$

$$\phi = 0 + \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \dots. \quad (34)$$

where  $x$  and  $t$  are function of  $\xi$  and  $\tau$  so partial derivatives with respect to  $x$  and  $t$  can be transform into partial derivative in terms of  $\xi$  and  $\tau$  so

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial x}, \Rightarrow \frac{\partial}{\partial x} = \varepsilon^{1/2} \frac{\partial}{\partial \xi} \quad (35)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial t}, \Rightarrow \frac{\partial}{\partial t} = -\varepsilon^{1/2} \frac{\partial}{\partial \xi} + \varepsilon^{3/2} \frac{\partial}{\partial \tau} \quad (36)$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x} \left( \varepsilon^{1/2} \frac{\partial}{\partial \xi} \right), \Rightarrow \frac{\partial^2}{\partial x^2} = \varepsilon \frac{\partial^2}{\partial \xi^2} \quad (37)$$

We can express (5)–(7) in terms of  $\xi$  and  $\tau$  as

$$\varepsilon^{3/2} \frac{\partial n_i}{\partial \tau} - \varepsilon^{1/2} \lambda \frac{\partial n_i}{\partial \xi} + \varepsilon^{1/2} \frac{\partial (n_i u_i)}{\partial \xi} = 0 \quad (38)$$

$$\varepsilon^{3/2} \frac{\partial u_i}{\partial \tau} - \varepsilon^{1/2} \lambda \frac{\partial u_i}{\partial \xi} + \varepsilon^{1/2} u_i \frac{\partial u_i}{\partial x} = -\varepsilon^{1/2} \frac{\partial \phi}{\partial x} \quad (39)$$

$$\varepsilon \frac{\partial^2 \phi}{\partial \xi^2} = e^\phi - n_i \quad (40)$$

Substituting the Eqs. (31)–(34) in Eqs. (38)–(40) and collecting the lowest order  $O(\varepsilon^{3/2})$  terms we get

$$-\lambda \frac{\partial n_i^{(1)}}{\partial \xi} + \frac{\partial u_i^{(1)}}{\partial \xi} = 0, \quad (41)$$

$$-\lambda \frac{\partial u_i^{(1)}}{\partial \xi} = \frac{\partial \phi^{(1)}}{\partial \xi}, \quad (42)$$

$$\phi^{(1)} - n_i^{(1)} = 0. \quad (43)$$

Integrating Eqs. (41)–(43) and all the variables tend to zero as  $\xi \rightarrow \infty$ . We get

$$n_i^{(1)} = \frac{u_i^{(1)}}{\lambda}, \quad (44)$$

$$u_i^{(1)} = \frac{\phi^{(1)}}{\lambda}, \quad (45)$$

$$\phi^{(1)} = n_i^{(1)}. \quad (46)$$

From Eq. (44)–(46) we get the phase velocity as

$$\lambda^2 = \pm 1 \quad (47)$$

Substituting the Eqs.(31)–(34) in Eqs. (38)–(40) and collecting order  $O(\varepsilon^{5/2})$ , we get

$$\frac{\partial n_i^{(1)}}{\partial \tau} - \lambda \frac{\partial n_i^{(2)}}{\partial \xi} + \frac{\partial n_i^{(1)} u_i^{(1)}}{\partial \xi} + \frac{\partial u_i^{(2)}}{\partial \xi} = 0, \quad (48)$$

$$\frac{\partial u_i^{(1)}}{\partial \tau} - \lambda \frac{\partial u_i^{(2)}}{\partial \xi} + u_i^{(1)} \frac{\partial u_i^{(1)}}{\partial \xi} = -\frac{\partial \phi^{(2)}}{\partial \xi^2}, \quad (49)$$

$$\frac{\partial \phi^{(1)}}{\partial \xi^2} = \phi^{(2)} + \frac{1}{2} \left( \phi^{(1)} \right)^2 - n_i^{(1)}. \quad (50)$$

Differentiating Eq. (50) With respect to  $\xi$  and substituting for  $\frac{\partial n_i^{(2)}}{\partial \xi}$  from Eq. (48) and for  $\frac{\partial u_i^{(2)}}{\partial \xi}$  from Eq. (49), we finally obtain

$$\frac{\partial \phi^{(1)}}{\partial \tau} + \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{1}{2} \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0. \quad (51)$$

Eq. (51) is known as **KdV equation**.  $\phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi}$  is the nonlinear term and  $\frac{1}{2} \frac{\partial^3 \phi^{(1)}}{\partial \xi^3}$  is the dispersive terms. Only nonlinearity can impose energy into the wave and the wave breaks but in presence of both nonlinearity and dispersive a stable wave profile is possible.

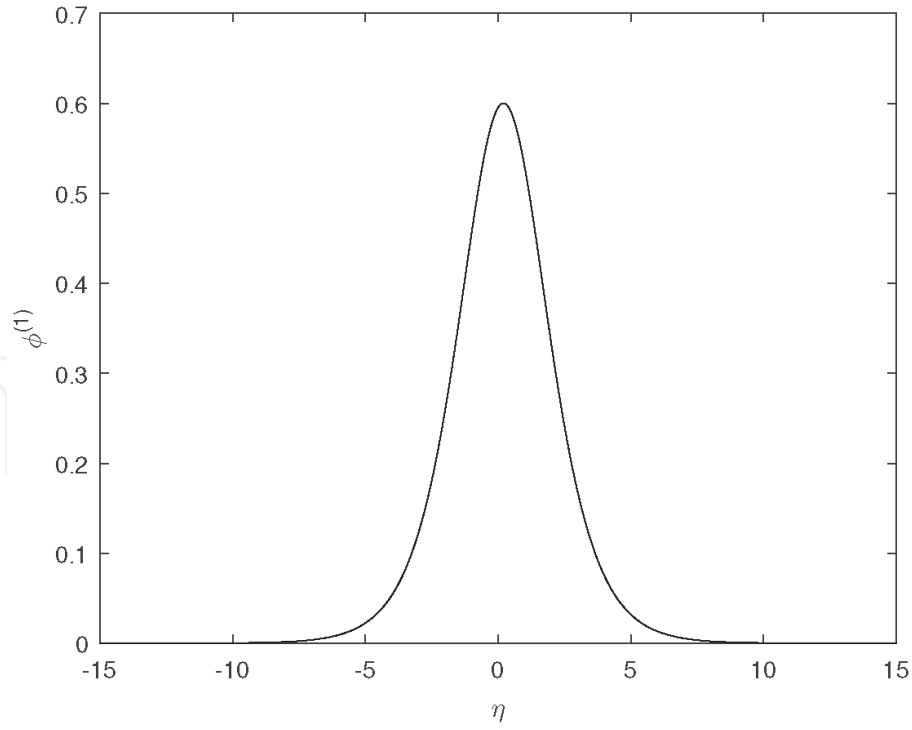
The steady-state solution of this KdV equation is obtained by transforming the independent variables  $\xi$  and  $\tau$  to  $\eta = \xi - u_0 \tau$  where  $u_0$  is a constant velocity normalized by  $c_s$ .

The steady state solution of the KdV Eq. (51) can be written as

$$\phi^{(1)} = \phi_m \operatorname{sech}^2 \left( \frac{\eta}{\Delta} \right) \quad (52)$$

where  $\phi_m = 3u_0$  and  $\Delta$  are the amplitude and width of the solitary waves. It is clear that height, width and speed of the pulse propotional to  $u_0$ ,  $\frac{1}{\sqrt{u_0}}$ , and  $u_0$  respectively. As  $\phi_m$  the amplitude is equal to  $3u_0$  so  $u_0$  specify the energy of the





**Figure 1.**  
Solitary wave solution of Eq. (52) for the parameter value  $t = 1, u_0 = 0.2$ .

solitary waves. So the larger the energy, the greater the speed, larger the amplitude and narrower the width (**Figure 1**).

### 3. Damped force KdV equation

Let us consider an unmagnetized collisional dusty plasma that contains cold inertial ions, stationary dusts with negative charge and Maxwellian electrons. The normalized ion fluid equations which include the equation of continuity, equation of momentum balance and Poisson's equation, governing the DIAWs, are given by

$$\frac{\partial n_i}{\partial t} + \frac{\partial(n_i u_i)}{\partial x} = 0, \quad (53)$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{\partial \phi}{\partial x} - \nu_{id} u_i, \quad (54)$$

$$\frac{\partial^2 \phi}{\partial x^2} = (1 - \mu)n_e - n + \mu, \quad (55)$$

where  $n_j$  ( $j = i, e$  for ion, electron),  $u_i, \phi$  are the number density, ion fluid velocity and the electrostatic wave potential respectively. Here  $\mu = \frac{Z_d n_{d0}}{n_0}$ ,  $\nu_{id}$  is the dust ion collisional frequency and the term  $S(x, t)$  [4, 5], is a charged density source arising from experimental conditions for a single definite purpose.  $n_0, Z_d, n_{d0}$  are the

#### 3.1 Normalization

$$n_i \rightarrow \frac{n_i}{n_0}, u_i \rightarrow \frac{u_i}{C_s}, \phi \rightarrow \frac{e\phi}{K_B T_e}, x \rightarrow \frac{x}{\lambda_D}, t \rightarrow \omega_{pi} t \quad (56)$$

where  $C_s \left( = \sqrt{\left( \frac{K_B T_e}{m_i} \right)} \right)$  is the ion acoustic speed,  $T_e$  as electron temperature,  $K_B$  as Boltzmann constant,  $e$  as magnitude of electron charge and  $m_i$  as mass of ions.  $\lambda_D \left( = \left( \frac{T_e}{4\pi n_e e^2} \right)^{\frac{1}{2}} \right)$  is the Debye length and  $\omega_{pi} \left( = \left( \frac{4\pi n_e e^2}{m_i} \right)^{\frac{1}{2}} \right)$  as ion-plasma frequency.

The normalized electron density is given by

$$n_e = e^\phi. \quad (57)$$

### 3.2 Phase velocity and nonlinear evolution equation

We introduced the same stretched coordinates use in Eq.(31). The expansion of the dependent variables also considered as (32)–(34) with

$$\nu_{id} \sim \varepsilon^{3/2} \nu_{id0}. \quad (58)$$

$$S \sim \varepsilon^2 S_2. \quad (59)$$

Substituting (31)–(34) and (58)–(59) along with stretching coordinates into Eqs. (53)–(55) and equating the coefficients of lowest order of  $\varepsilon$ , we get the phase velocity as

$$\lambda = \frac{1}{\sqrt{(1 - \mu)}}. \quad (60)$$

Taking the coefficients of next higher order of  $\varepsilon$ , we obtain the **damped force KdV equation**

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + B \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + C \phi^{(1)} = B \frac{\partial S_2}{\partial \xi}, \quad (61)$$

where  $A = \frac{3-\lambda^2}{2\lambda}$ ,  $B = \frac{\lambda^3}{2}$ ,  $C = \frac{\nu_{id0}}{2}$ .

It has been noticed that the behavior of nonlinear waves changes significantly in the presence of external periodic force. It is paramount to note that the source term or forcing term due to the presence of space debris in plasmas may be of different kind, for example, Gaussian forcing term [4], hyperbolic forcing term [4], (in the form of  $\text{sech}^2(\xi, \tau)$  and  $\text{sech}^4(\xi, \tau)$  functions) and trigonometric forcing term [6] (in the form of  $\sin(\xi, \tau)$  and  $\cos(\xi, \tau)$  functions). Motivated by these work we assume that  $S_2$  is a linear function of  $\xi$  such as  $S_2 = \frac{f_0 \xi}{B} \cos(\omega \tau) + P$ , where  $P$  is some constant and  $f_0, \omega$  denote the strength and the frequency of the source respectively. Put the expression of  $S_2$  in Eq. (61) we get,

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + B \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + C \phi^{(1)} = f_0 \cos(\omega \tau), \quad (62)$$

which is termed as **damped and forced KdV (DFKdV) equation**.

In absence of  $C$  and  $f_0$ , i.e., for  $C = 0$  and  $f_0 = 0$  the Eq.(62) takes the form of well-known KdV equation with the solitary wave solution

$$\phi_1 = \phi_m \text{sech}^2 \left( \frac{\xi - M\tau}{W} \right), \quad (63)$$

where  $\phi_m = \frac{3M}{A}$  and  $W = 2\sqrt{\frac{B}{M}}$ , with  $M$  as the Mach number.

In this case, it is well established that

$$I = \int_{-\infty}^{\infty} \phi_1^2 d\xi, \quad (64)$$

is a conserved. For small values of  $C$  and  $f_0$ , let us assume that the solution of Eq. (62) is of the form

$$\phi_1 = \phi_m(\tau) \operatorname{sech}^2 \left( \frac{x - M(\tau)\tau}{W(\tau)} \right), \quad (65)$$

where  $M(\tau)$  is an unknown function of  $\tau$  and  $\phi_m(\tau) = \frac{3M(\tau)}{A}$ ,  $W(\tau) = 2\sqrt{B/M(\tau)}$ . Differentiating Eq. (64) with respect to  $\tau$  and using Eq. (62), one can obtain

$$\begin{aligned} \frac{dI}{d\tau} + 2CI &= 2f_0 \cos(\omega\tau) \int_{-\infty}^{\infty} \phi_1 d\xi, \\ \Rightarrow \frac{dI}{d\tau} + 2CI &= \frac{24f_0\sqrt{B}}{A} \sqrt{M(\tau)} \cos(\omega\tau). \end{aligned} \quad (66)$$

Again,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \phi_1^2 d\xi, \\ I &= \int_{-\infty}^{\infty} \phi_m^2(\tau) \operatorname{sech}^4 \left( \frac{\xi - M(\tau)\tau}{W(\tau)} \right) d\xi, \\ I &= \frac{24\sqrt{B}}{A^2} M^{3/2}(\tau). \end{aligned} \quad (67)$$

Using Eq. (66) and (67) the expression of  $M(\tau)$  is obtained as

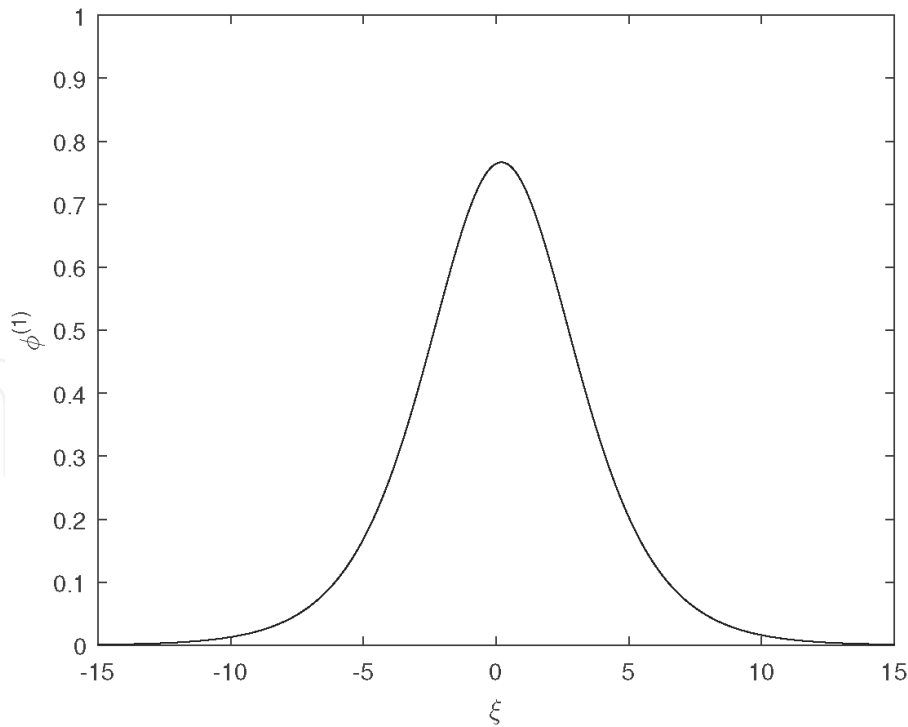
$$M(\tau) = \left( M - \frac{8ACf_0}{16C^2 + 9\omega^2} \right) e^{-\frac{4}{3}C\tau} + \frac{6Af_0}{16C^2 + 9\omega^2} \left( \frac{4}{3} C \cos(\omega\tau) + \omega \sin(\omega\tau) \right).$$

Therefore, the solution of the Eq. (62) is

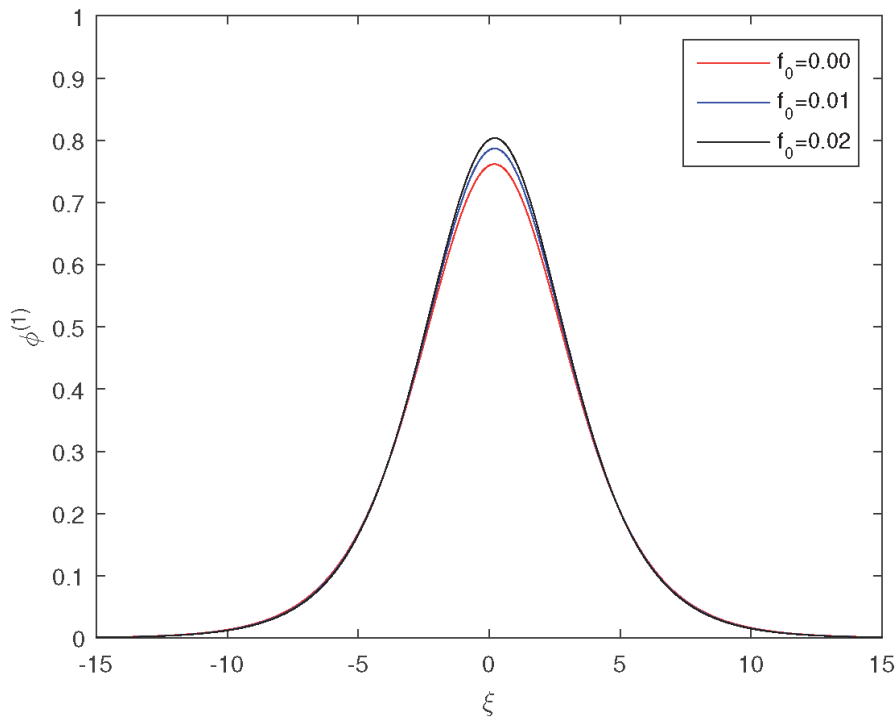
$$\phi_1 = \phi_m(\tau) \operatorname{sech}^2 \left( \frac{\xi - M(\tau)\tau}{W(\tau)} \right), \quad (68)$$

where  $\phi_m(\tau) = \frac{3M(\tau)}{A}$  and  $W(\tau) = 2\sqrt{\frac{B}{M(\tau)}}$ . The effect of the parameters, i.e., ion collision frequency parameter ( $\nu_{id0}$ ), strength of the external force ( $f_0$ ) on the solitary wave solution of the damp force KdV Eq. (62) have been numerically studied. In **Figure 2**, the soliton solution of (62) is plotted from (63) in the absence of external periodic force and damping.

In **Figure 3**, the soliton solution of the damp force KdV equation is plotted from Eq. (65) for different values of the strength of the external periodic force ( $f_0$ ). The values of other parameters are  $M_0 = 0.2$ ,  $\omega = 1$ ,  $\tau = 1$ ,  $\mu = 0.2$ ,  $\nu_{id0} = 0.01$ . It is observed that the solution produces solitary waves and the amplitude of the solitary waves increases as the value of the parameter  $f_0$  increases. In **Figure 4**, damp force KdV equation is plotted from Eq. (65) for different values of the dust ion collision

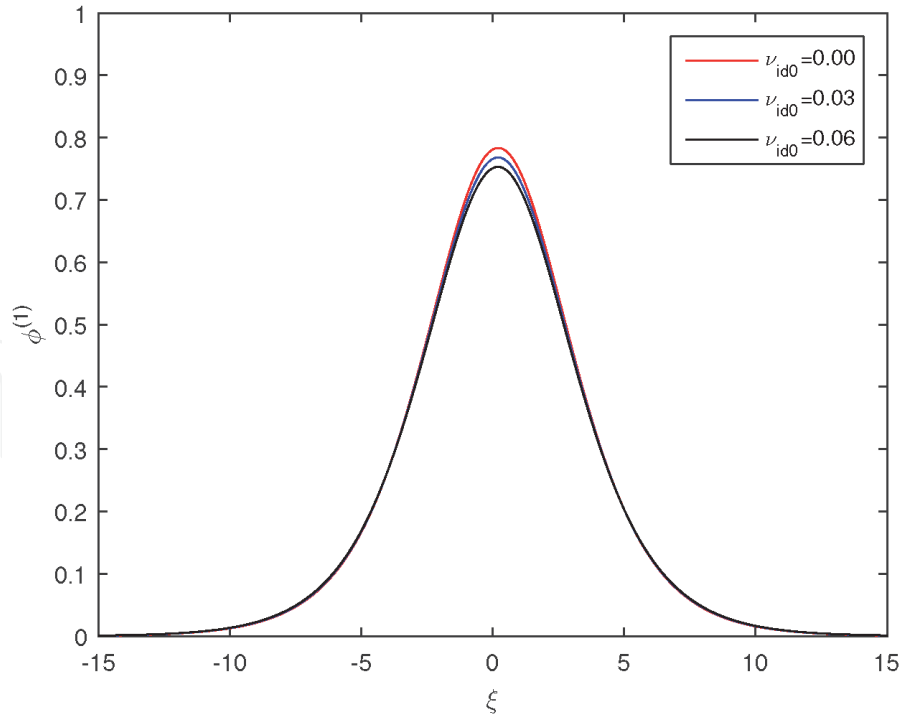


**Figure 2.**  
 Solitary wave solution of Eq. (62) in the absence of damping( $\nu_{id0} = 0$ ) and external force( $f_0 = 0$ ) with the parameter value  $M_0 = 0.2, \omega = 1, \tau = 1, \mu = 0.2$ .



**Figure 3.**  
 Variation of solitary wave from Eq. (62) for the different values of  $f_0$  with  $M_0 = 0.2, \omega = 1, \tau = 1, \mu = 0.2, \nu_{id0} = 0.01$ .

frequency parameter ( $\nu_{id0}$ ). The values of other parameters are  $M_0 = 0.2, \omega = 1, \tau = 1, \mu = 0.2, f_0 = 0.01$ . It is observed that the solution produces solitary waves and the amplitude of the solitary waves decreases as the value of the parameter  $\nu_{id0}$  increases and width of the solitary waves increases for increasing value of  $\nu_{id0}$ .



**Figure 4.** Variation of solitary wave from Eq. (62) for the different values of collisional frequency  $\nu_{id0}$  with  $M_o = 0.2$ ,  $\omega = 1$ ,  $\tau = 1$ ,  $\mu = 0.2$ ,  $f_o = 0.01$ .

#### 4. Damped KdV Burgers equation

To obtain damped KdV Burgers equation we considered an unmagnetized collisional dusty plasma which contains cold inertial ions, stationary dusts with negative charge and Maxwellian distributed electrons. The normalized ion fluid equations are as follows

$$\frac{\partial n_i}{\partial t} + \frac{\partial(n_i u_i)}{\partial x} = 0, \quad (69)$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{\partial \phi}{\partial x} + \eta \frac{\partial^2 u_i}{\partial x^2} - \nu_{id} u_i, \quad (70)$$

$$\frac{\partial^2 \phi}{\partial x^2} = (1 - \mu)n_e - n_i + \mu, \quad (71)$$

$$n_e = e^\phi, \quad (72)$$

where  $n_i, n_e, u_i, \phi$ , are the number density of ions, the number density of electrons, the ion fluid velocity and the electrostatic wave potential, respectively.

Here normalization is taken as follows

$$n_i \rightarrow \frac{n_i}{n_0}, u_i \rightarrow \frac{u_i}{C_s}, \phi \rightarrow \frac{e\phi}{K_B T_e}, x \rightarrow \frac{x}{\lambda_D}, t \rightarrow \omega_{pi} t$$

$C_s \left( = \sqrt{\left( \frac{K_B T_e}{m_i} \right)} \right)$  is the ion acoustic speed,  $T_e$  as electron temperature,  $K_B$  as Boltzmann constant and  $m_i$  as mass of ions,  $e$  as magnitude of electron charge.

$\lambda_D \left( = \left( \frac{T_e}{4\pi n_0 e^2} \right)^{\frac{1}{2}} \right)$  is the Debye length and  $\omega_{pi} \left( = \left( \frac{m_i}{4\pi n_0 e^2} \right)^{\frac{1}{2}} \right)$  as ion-plasma

frequency. Here,  $\nu_{id}$  is the dust-ion collisional frequency and  $\mu = \frac{n_{0e}}{n_{0i}}$ , where  $n_{0e}$  and  $n_{0i}$  are the unperturbed number densities of electrons and ions, respectively.

#### 4.1 Perturbation

To obtain damped KdV burger we introduced the same stretched coordinates use in Eq.(31). The expansion of the dependent variables are also considered same as (32)–(34) with

$$\eta = \varepsilon^{1/2} \eta_0, \quad (73)$$

$$\nu_{id} \sim \varepsilon^{3/2} \nu_{id0}. \quad (74)$$

#### 4.2 Phase velocity and nonlinear evolution equation

Substituting the above expansions (32)–(34) and (73)–(74) along with stretching coordinates (31) into Eqs. (69)–(71) and equating the coefficients of lowest order of  $\varepsilon$ , the phase velocity is obtained as

$$\lambda = \frac{1}{\sqrt{(1 - \mu)}}. \quad (75)$$

Taking the coefficients of next higher order of  $\varepsilon$ , we obtain the **DKdVB equation**

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + B \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + C \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + D \phi^{(1)} = 0, \quad (76)$$

where  $A = \frac{3-\lambda^2}{2\lambda}$ ,  $B = \frac{v^3}{2}$ ,  $C = -\frac{\eta_{10}}{2}$  and  $D = \frac{\nu_{id0}}{2}$ .

In absence of  $C$  and  $D$ , i.e., for  $C = 0$  and  $D = 0$  the Eq.(76) takes the form of well-known KdV equation with the solitary wave solution

$$\phi_1 = \phi_m \operatorname{sech}^2 \left( \frac{\xi - M_0 \tau}{W} \right), \quad (77)$$

where amplitude of the solitary waves  $\phi_m = \frac{3M_0}{A}$  and width of the solitary waves  $W = 2\sqrt{\frac{B}{M_0}}$ , with  $M_0$  is the speed of the ion-acoustic solitary waves or Mach number.

It is well established for the KdV equation that,

$$I = \int_{-\infty}^{\infty} \phi_1^2 d\xi, \quad (78)$$

is a conserved quantity [7].

For small values of  $C$  and  $D$ , let us assume that amplitude, width and velocity of the dust ion acoustic waves are dependent on  $\tau$  and the slow time dependent solution of Eq. (76) is of the form

$$\phi^{(1)} = \phi_m(\tau) \operatorname{sech}^2 \left( \frac{\xi - M(\tau)\tau}{W(\tau)} \right), \quad (79)$$

where the amplitude  $\phi_m(\tau) = \frac{3M(\tau)}{A}$ , width  $W(\tau) = 2\sqrt{B/M(\tau)}$  and velocity  $M(\tau)$  have to be determined.



Differentiating Eq. (78) with respect to  $\tau$  and using Eq. (76), one can obtain

$$\begin{aligned}\frac{dI}{d\tau} + 2DI &= 2C \int_{-\infty}^{\infty} \left( \frac{\partial \phi_1}{\partial \xi} \right)^2 d\xi, \\ \Rightarrow \frac{dI}{d\tau} + 2DI &= 2C \times \frac{24 M^{5/2}(\tau)}{5 A^2 \sqrt{B}}.\end{aligned}\quad (80)$$

where,

$$\int_{-\infty}^{\infty} \left( \frac{\partial \phi_1}{\partial \xi} \right)^2 d\xi = \frac{24 M^{5/2}(\tau)}{5 A^2 \sqrt{B}} \quad (81)$$

and

$$\begin{aligned}I &= \int_{-\infty}^{\infty} \phi_1^2 d\xi, \\ I &= \int_{-\infty}^{\infty} \phi_m^2(\tau) \operatorname{sech}^4 \left( \frac{\xi - M(\tau)\tau}{W(\tau)} \right) d\xi, \\ I &= \frac{24\sqrt{B}}{A^2} M^{3/2}(\tau).\end{aligned}\quad (82)$$

Substituting Eq. (81) and (82) into Eq. (80), we obtain

$$\frac{dM(\tau)}{d\tau} + PM(\tau) = QM^2(\tau), \quad (83)$$

which is the Bernoulli's equation, where  $P = \frac{4}{3}D$  and  $Q = \frac{4}{15}\frac{C}{B}$ .  
The solution of the Eq. (83) is

$$M(\tau) = \frac{PM_0}{M_0Q(1 - e^{P\tau}) + Pe^{P\tau}}$$

Therefore, the slow time dependence form of the ion acoustic solitary wave solution of the DKdVB Eq. (76) is given by (79) where.

$$M(\tau) = \frac{PM_0}{M_0Q(1 - e^{P\tau}) + Pe^{P\tau}} \text{ and } M(0) = M_0 \text{ for } \tau = 0.$$

## 5. Damped force MKdV equation

Let us consider an unmagnetized collisional dusty plasma that contains cold inertial ions, stationary dusts with negative charge and Maxwellian distributed electrons. The normalized ion fluid equations which include the equation of continuity, equation of momentum balance and Poisson's equation, governing the DIAWs, are given by

$$\frac{\partial n_i}{\partial t} + \frac{\partial(n_i u_i)}{\partial x} = 0, \quad (84)$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{\partial \phi}{\partial x} - \nu_{id} u_i, \quad (85)$$

$$\frac{\partial^2 \phi}{\partial x^2} = (1 - \mu)n_e - n_i + \mu + S(x, t) \quad (86)$$

where  $n_j$  ( $j = i, e$  for ion, electron),  $u_i$ ,  $\phi$  are the number density, ion fluid velocity and the electrostatic wave potential respectively. Here  $\mu = \frac{Z_d n_{d0}}{n_0}$ ,  $\nu_{id}$  is the dust-ion collisional frequency and the term  $S(x, t)$  [4, 5], is a charged density source arising from experimental conditions for a single definite purpose.  $n_0, Z_d, n_{d0}$  are the normalization:

$$n_i \rightarrow \frac{n_i}{n_0}, u_i \rightarrow \frac{u_i}{C_s}, \phi \rightarrow \frac{e\phi}{K_B T_e}, x \rightarrow \frac{x}{\lambda_D}, t \rightarrow \omega_{pi} t \quad (87)$$

where  $C_s = \left( \sqrt{\frac{K_B T_e}{m_i}} \right)$  is the ion acoustic speed,  $T_e$  as electron temperature,  $K_B$  as Boltzmann constant,  $e$  as magnitude of electron charge and  $m_i$  as mass of ions.  $\lambda_D = \left( \frac{T_e}{4\pi n_0 e^2} \right)^{\frac{1}{2}}$  is the Debye length and  $\omega_{pi} = \left( \frac{4\pi n_0 e^2}{m_i} \right)^{\frac{1}{2}}$  as ion-plasma frequency.

The normalized  $q$ -nonextensive electron number density takes the form [8]:

$$n_e = n_{e0} \{1 + (q - 1)\phi\}^{\frac{q+1}{2(q-1)}} \quad (88)$$

### Phase velocity and nonlinear evolution equation

We introduced the same stretched coordinates use in Eq. (31). The expansion of the dependent variables also considered same as (32)–(34) and (58)–(59). Substituting (31)–(34) and (58)–(59) along with stretching coordinates into Eqs. (84)–(86) and equating the coefficients of lowest order of  $\varepsilon$ , we get the phase velocity as

$$\lambda = \frac{1}{\sqrt{a(1 - \mu)}}, \quad (89)$$

with  $a = \frac{q+1}{2}$ . Now taking the coefficients of next higher order of  $\varepsilon$  [i.e., coefficient of  $\varepsilon^{5/2}$  from Eqs. (84) and (85) and coefficient of  $\varepsilon^2$  from Eq. (86)], we obtain the **DFKdV** equation

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + B \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + C \phi^{(1)} = B \frac{\partial S_2}{\partial \xi}, \quad (90)$$

where  $A = \left( \frac{3}{2\lambda} - \frac{b\lambda}{a} \right)$ ,  $B = \frac{\lambda^3}{2}$  and  $C = \frac{\nu_{id0}}{2}$ , with  $b = \frac{(q+1)(3-q)}{8}$ .

Now at the certain values, for example  $q = 0.6$  and  $\mu = 0.5$ , there is a critical point at which  $A = 0$ , which imply the infinite growth of the amplitude of the DIASW solution as nonlinearity goes to zero. Therefore, at the critical point at which  $A = 0$  the stretching (31) is not valid. For describing the evolution of the nonlinear system at or near the critical point we introduce the new stretched coordinate as

$$\xi = \varepsilon(x - \lambda t), \tau = \varepsilon^3 t, \quad (91)$$

and expand of the dependent variables same as Eqs. (32)–(34) with

$$\nu_{id} \sim \varepsilon^3 \nu_{id0}, \quad (92)$$

$$S \sim \varepsilon^3 S_2. \quad (93)$$

Now substituting Eq. (32)–(34) and (91)–(93) into the basic Eqs. (84)–(86) and equating the coefficients of lowest order of  $\varepsilon$ , [i.e., coefficients of  $\varepsilon^2$  from Eq. (84) and (85) and coefficients of  $\varepsilon$  from Eq. (86)], we obtain the following relations:

$$n_i^{(1)} = \frac{u_i^{(1)}}{\lambda}, \quad (94)$$

$$u_i^{(1)} = \frac{\phi^{(1)}}{\lambda}, \quad (95)$$

$$n_i^{(1)} = a(1 - \mu)\phi^{(1)}. \quad (96)$$

Equating the coefficients of next higher order of  $\varepsilon$ , [i.e., coefficients of  $\varepsilon^3$  from Eq. (84) and (85) and coefficients of  $\varepsilon$  from Eq. (86)], we obtain the following relations:

$$n_i^{(2)} = \frac{1}{\lambda} \left( u_i^{(2)} + n_i^{(1)} u_i^{(1)} \right) \quad (97)$$

$$\frac{\partial u_i^{(1)}}{\partial \xi} = \frac{1}{\lambda} \left( u_i^{(1)} \frac{\partial u_i^{(1)}}{\partial \xi} + \frac{\partial \phi^{(2)}}{\partial \xi} \right) \quad (98)$$

$$n_i^{(2)} = a(1 - \mu) \left( a\phi^{(2)} + b(\phi^{(1)})^2 \right) \quad (99)$$

Equating the coefficients of next higher order of  $\varepsilon$ , [i.e., coefficients of  $\varepsilon^4$  from Eq. (84) and (85) and coefficients of  $\varepsilon$  from Eq. (86)], we obtain the following relations:

$$\frac{\partial n_i^{(1)}}{\partial \tau} - \lambda \frac{\partial n_i^{(3)}}{\partial \xi} + \frac{\partial u_i^{(3)}}{\partial \xi} + \frac{\partial (n_i^{(1)} u_i^{(2)})}{\partial \xi} + \frac{\partial (n_i^{(2)} u_i^{(1)})}{\partial \xi} = 0 \quad (100)$$

$$\frac{\partial u_i^{(1)}}{\partial \tau} - \lambda \frac{\partial u_i^{(3)}}{\partial \xi} + \frac{\partial \phi^{(3)}}{\partial \xi} + \frac{\partial (u_i^{(1)} u_i^{(2)})}{\partial \xi} + \nu_{id0} u_i^{(1)} = 0 \quad (101)$$

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = (1 - \mu) \left( a\phi^{(3)} + 2b\phi^{(1)}\phi^{(2)} + c(\phi^{(1)})^3 \right) - n_i^{(3)} + S_2 \quad (102)$$

where  $a = \frac{(1+q)}{2}$ ,  $b = \frac{(1+q)(3-q)}{8}$  and  $c = \frac{(1+q)(3-q)(5-3q)}{48}$ .

From Eq. (94)–(96), one can obtain the Phase velocity as  $\lambda^2 = \frac{1}{a(1-\mu)}$  and from Eqs. (94)–(102), one can obtain the following nonlinear evaluation equation as:

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A_1 (\phi^{(1)})^2 \frac{\partial \phi^{(1)}}{\partial \xi} + B_1 \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + C_1 \phi^{(1)} = B_1 \frac{\partial S_2}{\partial \xi}, \quad (103)$$

where  $A_1 = \frac{15}{4\lambda^3} - \frac{3\lambda^3 c(1-\mu)}{2}$ ,  $B_1 = \frac{\lambda^3}{2}$  and  $C_1 = \frac{\nu_{id0}}{2}$ .

It has been noticed that the behavior of nonlinear waves changes significantly in the presence of external periodic force. For simplicity, we assume that  $S_2$  is a linear function of  $\xi$  such as  $S_2 = f_0 \xi \cos(\omega\tau) + P$ , where  $P$  is some constant and  $f_0$ ,  $\omega$  denote the strength and the frequency of the source respectively. Put the expression of  $S_2$  in the Eq. (103) we get,

$$\frac{\partial \phi^{(1)}}{\partial \tau} + A_1 \left( \phi^{(1)} \right)^2 \frac{\partial \phi^{(1)}}{\partial \xi} + B_1 \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + C_1 \phi^{(1)} = B_1 f_0 \cos(\omega \tau). \quad (104)$$

Such a form of this source function is observed in experimental situations or conditions for a particular device. Eq. (104) is termed as **damped force modified Korteweg-de Varies (DFMKdV)** equation.

In absence of  $C_1$  and  $f_0$ , i.e., for  $C_1 = 0$  and  $f_0 = 0$  the Eq.(104) takes the form of well-known **MKdV equation**.

The slow time dependence form of the ion acoustic waves solution of the DFMKdV Eq. (104) is given by,

$$\phi^{(1)} = \phi_m(\tau) \operatorname{sech} \left( \frac{\xi - M(\tau)\tau}{W(\tau)} \right), \quad (105)$$

where  $M(\tau)$  is given by equation

$$M(\tau) = \left[ \frac{\pi f_0 B_1 \sqrt{A_1/6}}{2} \left( \frac{\omega}{\omega^2 + 4C_1^2} \right) \left\{ \sin(\omega \tau) + \frac{2C_1}{\omega} \cos(\omega \tau) \right\} + \left\{ \sqrt{M} - \pi f_0 B_1 \sqrt{A_1/24} \left( \frac{2C_1}{\omega^2 + 4C_1^2} \right) \right\} e^{-\nu_{id0}\tau} \right]^2.$$

The amplitude and width are as follows:

$$\phi_m(\tau) = \frac{1}{\sqrt{A}} \left( \sqrt{6} \left[ \frac{\pi f_0 B_1 \sqrt{A_1/6}}{2} \left( \frac{\omega}{\omega^2 + 4C_1^2} \right) \left\{ \sin(\omega \tau) + \frac{2C_1}{\omega} \cos(\omega \tau) \right\} + \left\{ \sqrt{M} - \pi f_0 B_1 \sqrt{A_1/24} \left( \frac{2C_1}{\omega^2 + 4C_1^2} \right) \right\} e^{-\nu_{id0}\tau} \right] \right) \\ W(\tau) = \frac{\sqrt{B_1}}{W_1 + W_2}$$

where

$$W_1 = \frac{\pi f_0 B_1 \sqrt{A_1/6}}{2} \left( \frac{\omega}{\omega^2 + 4C_1^2} \right) \left\{ \sin(\omega \tau) + \frac{2C_1}{\omega} \cos(\omega \tau) \right\} \\ W_2 = \left\{ \sqrt{M} - \pi f_0 B_1 \sqrt{A_1/24} \left( \frac{2C_1}{\omega^2 + 4C_1^2} \right) \right\} e^{-\nu_{id0}\tau}$$

## 6. Damped force Zakharov-Kuznetsov equation

Let us consider a plasma model [9] consisting of cold ions, Maxwellian electrons in the presence of dust particles and the external static magnetic field  $B = \hat{y}B_0$  along the y-axis. The normalized continuity, momentum and Poisson's equations are as follows

$$\frac{\partial n}{\partial t} + \frac{\partial(nu)}{\partial x} + \frac{\partial(nv)}{\partial y} + \frac{\partial(nw)}{\partial z} = 0, \quad (106)$$

$$\frac{\partial u}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) u = -\frac{\partial \phi}{\partial x} - \frac{\Omega_i}{\omega_{pi}} w, \quad (107)$$

$$\frac{\partial v}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) v = -\frac{\partial \phi}{\partial y} - \nu_{id} v, \quad (108)$$

$$\frac{\partial w}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) w = -\frac{\partial \phi}{\partial z} + \frac{\Omega_i}{\omega_{pi}} u, \quad (109)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \delta_1 + \delta_2 n_e - n \quad (110)$$

The normalized electron density is given by

$$n_e = e^\phi, \quad (111)$$

where  $n, n_e, u_i (= u, v, w), T_e, m_i, e, \phi, \Omega_i, \omega_{pi}, \nu_{id}$  and  $\lambda_D$  are the ion number density, electron number density, ion velocity, electron temperature, ion mass, electron charge, electrostatic potential, ion cyclotron frequency, ion plasma frequency, dust ion collision frequency and Debye length respectively.

Here the normalization is done as follows:

$$n \rightarrow \frac{n}{n_0}, n_e \rightarrow \frac{n_e}{n_{e0}}, u_i \rightarrow \frac{u_i}{C_s}, \phi \rightarrow \frac{e\phi}{T_e}, x \rightarrow \frac{x}{\lambda_D}, t \rightarrow \omega_{pi} t$$

Here  $\delta_1 = \frac{n_{d0}}{n_{i0}}, \delta_2 = \frac{n_{e0}}{n_{i0}}$  with the condition  $\delta_1 + \delta_2 = 1$ .  $\lambda_D = \left( \frac{T_e}{4\pi n_{e0} e^2} \right)^{1/2}$ ,  $\omega_{pi}^{-1} = \left( \frac{m_i}{4\pi n_{e0} e^2} \right)^{1/2}, C_s = \sqrt{\frac{T_e}{m_i}}$ .

To obtain the DFZK equation we introduce the new stretched coordinates as

$$\begin{aligned} \xi &= \varepsilon^{1/2} x \\ \zeta &= \varepsilon^{1/2} (x - \lambda t), \\ \eta &= \varepsilon^{1/2} y, \\ \tau &= \varepsilon^{3/2} t \end{aligned} \quad (112)$$

where  $\varepsilon$  is the strength of nonlinearity and  $\lambda$  is the phase velocity of waves. The expression of the dependent variables as follows:

$$n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \dots \quad (113)$$

$$u = 0 + \varepsilon^{3/2} u_1 + \varepsilon^2 u_2 + \dots \quad (114)$$

$$v = 0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots \quad (115)$$

$$w = 0 + \varepsilon^{3/2} w_1 + \varepsilon^2 w_2 + \dots \quad (116)$$

$$\phi = 0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \dots \quad (117)$$

$$\nu_{id} \sim \varepsilon^{3/2} \nu_{id0} \quad (118)$$

$$S(x, y, z) \sim \varepsilon^2 S_2(x, y, z) \quad (119)$$

Substituting the equations (112)-(119) into the system of Eqs. (106)-(110) equating the coefficient of  $\varepsilon$ , we get

$$\phi_1 = \frac{n_1}{\delta_2}. \quad (120)$$

Equating the coefficient of  $\varepsilon^{3/2}$ , we get

$$n_1 = \frac{v_1}{\lambda} \quad (121)$$

$$w_1 = -\frac{\omega_{pi}}{\Omega_i} \frac{\partial \phi_1}{\partial \xi}, \quad (122)$$

$$v_1 = \frac{\phi_1}{\lambda}, \quad (123)$$

$$u_1 = \frac{\omega_{pi}}{\Omega_i} \frac{\partial \phi_1}{\partial \eta}. \quad (124)$$

Considering the coefficient of  $\varepsilon^2$ , the following relationships are obtained

$$w_2 = \lambda \frac{\omega_{pi}}{\Omega_i} \frac{\partial u_1}{\partial \zeta}, \quad (125)$$

$$u_2 = -\lambda \frac{\omega_{pi}}{\Omega_i} \frac{\partial w_1}{\partial \zeta}, \quad (126)$$

$$\frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{\partial^2 \phi_1}{\partial \zeta^2} + \frac{\partial^2 \phi_1}{\partial \eta^2} = \delta_1 \left( 1 - \phi_2 + \frac{\phi_1^2}{2} \right) - n_2 + S_2. \quad (127)$$

Comparing the coefficients of  $\varepsilon^{5/2}$ , we obtain

$$\frac{\partial n_1}{\partial \tau} - \lambda \frac{\partial n_2}{\partial \zeta} + \frac{\partial u_2}{\partial \xi} + \frac{\partial}{\partial \zeta} (n_1 v_1) + \frac{\partial v_2}{\partial \zeta} + \frac{\partial w_2}{\partial \eta} = 0 \quad (128)$$

$$\frac{\partial v_1}{\partial \tau} - \lambda \frac{\partial v_2}{\partial \zeta} + v_1 \frac{\partial v_1}{\partial \zeta} + \frac{\partial \phi_2}{\partial \zeta} - \nu_{id0} v_1 = 0. \quad (129)$$

Using the relationships (120)–(124), one can obtain the linear dispersion relation as

$$1 - \lambda^2 \delta_2 = 0 \quad (130)$$

Expressing all the perturbed quantities in terms of  $\phi_1$  from Eq. (125)–(129), the **damped forced ZK equation** is obtained as

$$\frac{\partial \phi_1}{\partial \tau} + A \phi_1 \frac{\partial \phi_1}{\partial \zeta} + B \frac{\partial^3 \phi_1}{\partial \zeta^3} + D \phi_1 + C \frac{\partial}{\partial \zeta} \left( \frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{\partial^2 \phi_1}{\partial \eta^2} \right) + B \frac{\partial S_2}{\partial \zeta} = 0 \quad (131)$$

where

$$A = \frac{3}{2\lambda} - \frac{\lambda}{2}, \quad B = \frac{\lambda}{2\delta_2}, \quad C = \frac{\lambda}{2\delta_2} \left( 1 + \frac{\omega_{pi}^2}{\Omega_i^2} \right), \quad D = \frac{\nu_{id0}}{2}.$$

Choudhury et al. [5] studied analytical electron acoustic solitary wave (EASW) solution in the presence of periodic force for an unmagnetized plasma consisting of cold electron fluid, superthermal hot electrons and stationary ions. Motivated by the these works, here we consider the source term as  $S_2 = \frac{f_0}{B} (e\zeta + f\xi + g\eta) \cos(\omega\tau)$ ,



where  $f_0$  and  $\omega$  denote the strength and frequency of the source term respectively. Then Eq. (131) is of the form,

$$\frac{\partial \phi_1}{\partial \tau} + A\phi_1 \frac{\partial \phi_1}{\partial \zeta} + B \frac{\partial^3 \phi_1}{\partial \zeta^3} + D\phi_1 + C \frac{\partial}{\partial \zeta} \left( \frac{\partial^2 \phi_1}{\partial \zeta^2} + \frac{\partial^2 \phi_1}{\partial \eta^2} \right) = F_0 \cos(\omega\tau) \quad (132)$$

where  $F_0 = -\frac{ef_0}{B}$ . To find the analytical solution of Eq. (132), we transform the damped-forced ZK equation to the KdV equation. We introduce new variable:

$$\xi = (l\zeta + m\xi + n\eta), \quad (133)$$

where  $l, m, n$  are the direction cosines of the line of wave propagation, with  $l^2 + m^2 + n^2 = 1$ . Substituting Eqs. (133) into the Eq. (132), we get

$$\begin{aligned} \frac{\partial \phi_1}{\partial \tau} + Al\phi_1 \frac{\partial \phi_1}{\partial \xi} + Bl^3 \frac{\partial^3 \phi_1}{\partial \xi^3} + Cl(m^2 + n^2) \frac{\partial^3 \phi_1}{\partial \xi^3} + D\phi_1 &= F_0 \cos(\omega\tau) \\ \Rightarrow \frac{\partial \phi_1}{\partial \tau} + P\phi_1 \frac{\partial \phi_1}{\partial \xi} + Q \frac{\partial^3 \phi_1}{\partial \xi^3} + D\phi_1 &= F_0 \cos(\omega\tau) \end{aligned} \quad (134)$$

where,  $P = Al$ ,  $Q = Bl^3 + Cl(m^2 + n^2)$ ,

The analytical solitary wave solution of the Eq. (134) as obtained in (68), is

$$\phi_1 = \phi_m(\tau) \operatorname{sech}^2 \left( \frac{\xi - M(\tau)\tau}{W(\tau)} \right) \quad (135)$$

where  $\phi_m(\tau) = \frac{3M(\tau)}{P}$  and  $W(\tau) = 2\sqrt{\frac{Q}{M(\tau)}}$ , with

$$M(\tau) = \left( M - \frac{8PF_0}{16D^2 + 9\omega^2} \right) e^{-\frac{4}{3}D\tau} + \frac{6PF_0}{16D^2 + 9\omega^2} \left( \frac{4}{3}D\cos(\omega\tau) + \omega\sin(\omega\tau) \right). \quad (136)$$

## 7. Conclusions

It is clear from the structure of the solitary wave solution of the DFKdV, DFMKdV and DFZK that the soliton amplitude and width depends on the nonlinearity and dispersion of the evolution equations, which are the function of different plasma parameter involve in the consider plasma system. Also evident from the structure of the approximate analytical solution that the amplitude and the width of the soliton depends on the Mach number ( $M(\tau)$ ) which involve the forcing term  $F_0 \cos(\omega\tau)$  and the damping parameter. Thus the amplitude and the width of the solitary wave structure changes with the different plasma parameters. Also they are changes with the change of strength of external force  $F_0$ , frequency of the external force  $\omega$  and the collisional frequency between the different plasma species. The effect of these parameter can be studied through numerical simulation.

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